

SMALL FEEDBACK VERTEX SETS IN PLANAR DIGRAPHS

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ABSTRACT. Let G be a directed planar graph on n vertices, with no directed cycle of length less than $g \geq 4$. We prove that G contains a set X of vertices such that $G - X$ has no directed cycle, and $|X| \leq \frac{5n-5}{9}$ if $g = 4$, $|X| \leq \frac{2n-5}{4}$ if $g = 5$, and $|X| \leq \frac{2n-6}{g}$ if $g \geq 6$. This improves recent results of Golowich and Rolnick.

A directed graph G (or digraph, in short) is said to be acyclic if it does not contain any directed cycle. The *digirth* of a digraph G is the minimum length of a directed cycle in G (if G is acyclic, we set its digirth to $+\infty$). A *feedback vertex set* in a digraph G is a set X of vertices such that $G - X$ is acyclic, and the minimum size of such a set is denoted by $\tau(G)$. In this short note, we study the maximum $f_g(n)$ of $\tau(G)$ over all planar digraphs G on n vertices with digirth g . Harutyunyan [1, 4] conjectured that $f_3(n) \leq \frac{2n}{5}$ for all n . This conjecture was recently refuted by Knauer, Valicov and Wenger [5] who showed that $f_g(n) \geq \frac{n-1}{g-1}$ for all $g \geq 3$ and infinitely many values of n . On the other hand, Golowich and Rolnick [3] recently proved that $f_4(n) \leq \frac{7n}{12}$, $f_5(n) \leq \frac{8n}{15}$, and $f_g(n) \leq \frac{3n-6}{g}$ for all $g \geq 6$ and n . Harutyunyan and Mohar [4] proved that the vertex set of every planar digraph of digirth at least 5 can be partitioned into two acyclic subgraphs. This result was very recently extended to planar digraphs of digirth 4 by Li and Mohar [6], and therefore $f_4(n) \leq \frac{n}{2}$.

This short note is devoted to the following result, which improves all the previous upper bounds for $g \geq 5$ (although the improvement for $g = 5$ is rather minor). Due to the very recent result of Li and Mohar [6], our result for $g = 4$ is not best possible (however its proof is of independent interest and might lead to further improvements).

Theorem 1. *For all $n \geq 3$ we have $f_4(n) \leq \frac{5n-5}{9}$, $f_5(n) \leq \frac{2n-5}{4}$ and for all $g \geq 6$, $f_g(n) \leq \frac{2n-6}{g}$.*

In a planar graph, the degree of a face F , denoted by $d(F)$, is the sum of the lengths (number of edges) of the boundary walks of F . In the proof of Theorem 1, we will need the following two simple lemmas.

Lemma 2. *Let H be a planar bipartite graph, with bipartition (U, V) , such that all faces of H have degree at least 4, and all vertices of V have degree at least 2. Then H contains at most $2|U| - 4$ faces of degree at least 6.*

Proof. Assume that H has n vertices, m edges, f faces, and f_6 faces of degree at least 6. Let N be the sum of the degrees of the faces of H , plus

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twice the sum of the degrees of the vertices of V . Observe that $N = 4m$, so, by Euler's formula, $N \leq 4n + 4f - 8$. The sum of degrees of the faces of H is at least $4(f - f_6) + 6f_6 = 4f + 2f_6$, and since each vertex of V has degree at least 2, the sum of the degrees of the vertices of V is at least $2|V|$. Therefore, $4f + 2f_6 + 4|V| \leq 4n + 4f - 8$. It follows that $f_6 \leq 2|U| - 4$, as desired. \square

Lemma 3. *Let G be a connected planar graph, and let $S = \{F_1, \dots, F_k\}$ be a set of k faces of G , such that each F_i is bounded by a cycle, and these cycles are pairwise vertex-disjoint. Then $\sum_{F \notin S} (3d(F) - 6) \geq \sum_{i=1}^k (3d(F_i) + 6) - 12$, where the first sum varies over faces F of G not contained in S .*

Proof. Let n , m , and f denote the number of vertices, edges, and faces of G , respectively. It follows from Euler's formula that the sum of $3d(F) - 6$ over all faces of G is equal to $6m - 6f = 6n - 12 \geq 6 \sum_{i=1}^k d(F_i) - 12$. Therefore, $\sum_{F \notin S} (3d(F) - 6) \geq 6 \sum_{i=1}^k d(F_i) - 12 - \sum_{i=1}^k (3d(F_i) - 6) = \sum_{i=1}^k (3d(F_i) + 6) - 12$, as desired. \square

We are now able to prove Theorem 1.

Proof of Theorem 1. We prove the result by induction on $n \geq 3$. Let G be a planar digraph with n vertices and digirth $g \geq 4$. We can assume without loss of generality that G has no multiple arcs, since $g \geq 4$ and removing one arc from a collection of multiple arcs with the same orientation does not change the value of $\tau(G)$. We can also assume that G is connected, since otherwise we can consider each connected component of G separately and the result clearly follows from the induction (since $g \geq 4$, connected components of at most 2 vertices are acyclic and can thus be left aside). Finally, we can assume that G contains a directed cycle, since otherwise $\tau(G) = 0 \leq \min\{\frac{5n-5}{9}, \frac{2n-5}{4}, \frac{2n-6}{g}\}$ (since $n \geq 3$).

Let \mathcal{C} be a maximum collection of arc-disjoint directed cycles in G . Note that \mathcal{C} is non-empty. Fix a planar embedding of G . For a given directed cycle C of \mathcal{C} , we denote by \overline{C} the closed region bounded by C , and by $\overset{\circ}{C}$ the interior of \overline{C} . It follows from classical uncrossing techniques (see [2] for instance), that we can assume without loss of generality that the directed cycles of \mathcal{C} are pairwise non-crossing, i.e. for any two elements $C_1, C_2 \in \mathcal{C}$, either $\overset{\circ}{C}_1$ and $\overset{\circ}{C}_2$ are disjoint, or one is contained in the other. We define the partial order \preceq on \mathcal{C} as follows: $C_1 \preceq C_2$ if and only if $\overset{\circ}{C}_1 \subseteq \overset{\circ}{C}_2$. Note that \preceq naturally defines a rooted forest \mathcal{F} with vertex set \mathcal{C} : the roots of each of the components of \mathcal{F} are the maximal elements of \preceq , and the children of any given node $C \in \mathcal{F}$ are the maximal elements $C' \preceq C$ distinct from C (the fact that \mathcal{F} is indeed a forest follows from the non-crossing property of the elements of \mathcal{C}).

Consider a node C of \mathcal{F} , and the children C_1, \dots, C_k of C in \mathcal{F} . We define the closed region $\mathcal{R}_C = \overline{C} - \bigcup_{1 \leq i \leq k} \overset{\circ}{C}_i$. Let ϕ_C be the sum of $3d(F) - 6$, over all faces F of G lying in \mathcal{R}_C .

Claim 4. *Let C_0 be a node of \mathcal{F} with children C_1, \dots, C_k . Then $\phi_{C_0} \geq \frac{3}{2}(g-2)k + \frac{3}{2}g$. Moreover, if $g \geq 6$, then $\phi_{C_0} \geq \frac{3}{2}(g-2)k + \frac{3}{2}g + 3$.*

Assume first that the cycles C_0, \dots, C_k are pairwise vertex-disjoint. Then, it follows from Lemma 3 that $\phi_{C_0} \geq (k+1)(3g+6) - 12$. Note that since $g \geq 4$, we have $(k+1)(3g+6) - 12 \geq \frac{3}{2}(g-2)k + \frac{3}{2}g$. Moreover, if $g \geq 6$, $(k+1)(3g+6) - 12 \geq \frac{3}{2}(g-2)k + \frac{3}{2}g + 3$, as desired. As a consequence, we can assume that two of the cycles C_0, \dots, C_k intersect, and in particular, $k \geq 1$.

Consider the following planar bipartite graph H : the vertices of the first partite set of H are the directed cycles C_0, C_1, \dots, C_k , the vertices of the second partite set of H are the vertices of G lying in at least two cycles among C_0, C_1, \dots, C_k , and there is an edge in H between some cycle C_i and some vertex v if and only if $v \in C_i$ in G (see Figure 1). Observe that H has a natural planar embedding in which all internal faces have degree at least 4. Since $k \geq 1$ and at least two of the cycles C_0, \dots, C_k intersect, the outerface also has degree at least 4. Note that the faces F_1, \dots, F_t of H are in one-to-one correspondence with the maximal subsets $\mathcal{D}_1, \dots, \mathcal{D}_t$ of \mathcal{R}_{C_0} whose interior is connected. Also note that each face of $G \cap \mathcal{R}_{C_0}$ is in precisely one region \mathcal{D}_i and each arc of $\bigcup_{i=0}^k C_i$ (i.e. each arc on the boundary of \mathcal{R}_{C_0}) is on the boundary of precisely one region \mathcal{D}_i . For each region \mathcal{D}_i , let ℓ_i be the number of arcs on the boundary of \mathcal{D}_i , and observe that $\sum_{i=1}^t \ell_i = \sum_{j=0}^k |C_j|$. Let $\phi_{\mathcal{D}_i}$ be the sum of $3d(F) - 6$, over all faces F of G lying in \mathcal{D}_i . It follows from Lemma 3 (applied with $k = 1$) that $\phi_{\mathcal{D}_i} \geq 3\ell_i - 6$, and therefore $\phi_{C_0} = \sum_{i=1}^t \phi_{\mathcal{D}_i} \geq \sum_{i=1}^t (3\ell_i - 6)$.

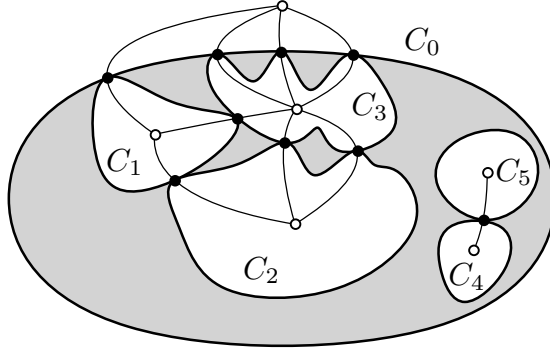


FIGURE 1. The region \mathcal{R}_{C_0} (in gray) and the planar bipartite graph H .

A region \mathcal{D}_i with $\ell_i \geq 4$ is said to be of *type 1*, and we set $T_1 = \{1 \leq i \leq t \mid \mathcal{D}_i \text{ is of type 1}\}$. Since for any $\ell \geq 4$ we have $3\ell - 6 \geq \frac{3\ell}{2}$, it follows from the paragraph above that the regions \mathcal{D}_i of type 1 satisfy $\phi_{\mathcal{D}_i} \geq \frac{3\ell_i}{2}$. Let \mathcal{D}_i be a region that is not of type 1. Since G is simple, $\ell_i = 3$. Assume first that \mathcal{D}_i is bounded by (parts of) two directed cycles of \mathcal{C} (in other words, \mathcal{D}_i corresponds to a face of degree four in the graph H). In this case we say that \mathcal{D}_i is of *type 2* and we set $T_2 = \{1 \leq i \leq t \mid \mathcal{D}_i \text{ is of type 2}\}$. Then the boundary of \mathcal{D}_i consists in two consecutive arcs e_1, e_2 of some directed cycle C^+ of \mathcal{C} , and one arc e_3 of some directed cycle C^- of \mathcal{C} . Since $g \geq 4$, these three arcs do not form a directed cycle, and therefore their orientation is transitive. It follows that $|C^+| \geq g + 1$, since otherwise

the directed cycle obtained from C^+ by replacing e_1, e_2 with e_3 would have length $g - 1$, contradicting that G has digirth at least g . Consequently, $\sum_{i=0}^k |C_i| \geq (k+1)g + |T_2|$. If a region \mathcal{D}_i is not of type 1 or 2, then $\ell_i = 3$ and each of the 3 arcs on the boundary of \mathcal{D}_i belongs to a different directed cycle of \mathcal{C} . In other words, \mathcal{D}_i corresponds to some face of degree 6 in the graph H . Such a region \mathcal{D}_i is said to be of *type 3*, and we set $T_3 = \{1 \leq i \leq t \mid \mathcal{D}_i \text{ is of type 3}\}$. It follows from Lemma 2 that the number of faces of degree at least 6 in H is at most $2(k+1) - 4$. Hence, we have $|T_3| \leq 2k - 2$.

Using these bounds on $|T_2|$ and $|T_3|$, together with the fact that for any $i \in T_2 \cup T_3$ we have $\phi_{\mathcal{D}_i} \geq 3\ell_i - 6 = 3 = \frac{3\ell_i}{2} - \frac{3}{2}$, we obtain:

$$\begin{aligned} \phi_{C_0} &= \sum_{i \in T_1} \phi_{\mathcal{D}_i} + \sum_{i \in T_2} \phi_{\mathcal{D}_i} + \sum_{i \in T_3} \phi_{\mathcal{D}_i} \\ &\geq \sum_{i=1}^t \left(\frac{3\ell_i}{2} - \frac{3}{2} \right) |T_2| - \frac{3}{2} |T_3| \\ &\geq \frac{3}{2} \sum_{i=0}^k |C_i| - \frac{3}{2} |T_2| - \frac{3}{2} (2k - 2) \\ &\geq \frac{3}{2} (k+1)g - 3k + 3 = \frac{3}{2} (g-2)k + \frac{3}{2}g + 3, \end{aligned}$$

as desired. This concludes the proof of Claim 4. \square

Let C_1, \dots, C_{k_∞} be the k_∞ maximal elements of \preceq . We denote by \mathcal{R}_∞ the closed region obtained from the plane by removing $\bigcup_{i=1}^{k_\infty} \mathring{C}_i$. Note that each face of G lies in precisely one of the regions \mathcal{R}_C ($C \in \mathcal{C}$) or \mathcal{R}_∞ . Let ϕ_∞ be the sum of $3d(F) - 6$, over all faces F of G lying in \mathcal{R}_∞ . A proof similar to that of Claim 4 shows that $\phi_\infty \geq \frac{3}{2}k_\infty(g-2) + 3$, and if $g \geq 6$, then $\phi_\infty \geq \frac{3}{2}k_\infty(g-2) + 6$.

We now compute the sum ϕ of $3d(F) - 6$ over all faces F of G . By Claim 4,

$$\begin{aligned} \phi &= \phi_\infty + \sum_{C \in \mathcal{F}} \phi_C \\ &\geq \frac{3}{2}k_\infty(g-2) + 3 + (|\mathcal{C}| - k_\infty)\frac{3}{2}(g-2) + |\mathcal{C}| \cdot \frac{3}{2}g \\ &\geq (3g-3)|\mathcal{C}| + 3. \end{aligned}$$

If $g \geq 6$, a similar computation gives $\phi \geq 3g|\mathcal{C}| + 6$. On the other hand, it easily follows from Euler's formula that $\phi = 6n - 12$. Therefore, $|\mathcal{C}| \leq \frac{2n-5}{g-1}$, and if $g \geq 6$, then $|\mathcal{C}| \leq \frac{2n-6}{g}$.

Let A be a set of arcs of G of minimum size such that $G - A$ is acyclic. It follows from the Lucchesi-Younger theorem [7] (see also [3]) that $|A| = |\mathcal{C}|$. Let X be a set of vertices covering the arcs of A , such that X has minimum size. Then $G - X$ is acyclic. If $g = 5$ we have $|X| \leq |A| = |\mathcal{C}| \leq \frac{2n-5}{4}$ and if $g \geq 6$, we have $|X| \leq |A| = |\mathcal{C}| \leq \frac{2n-6}{g}$, as desired. Assume now that $g = 4$. In this case $|A| = |\mathcal{C}| \leq \frac{2n-5}{3}$. It was observed by Golowich and Rolnick [3] that $|X| \leq \frac{1}{3}(n + |A|)$ (which easily follows from the fact that any graph on

n vertices and m edges contains an independent set of size at least $\frac{2n}{3} - \frac{m}{3}$, and thus, $|X| \leq \frac{5n-5}{9}$. This concludes the proof of Theorem 1. \square

FINAL REMARK

A natural problem is to determine the precise value of $f_g(n)$, or at least its asymptotical value as g tends to infinity. We believe that $f_g(n)$ should be closer to the lower bound of $\frac{n-1}{g}$, than to our upper bound of $\frac{2n-6}{g}$.

For a digraph G , let $\tau^*(G)$ denote the the infimum real number x for which there are weights in $[0, 1]$ on each vertex of G , summing up to x , such that for each directed cycle C , the sum of the weights of the vertices lying on C is at least 1. Goemans and Williamson [2] conjectured that for any planar digraph G , $\tau(G) \leq \frac{3}{2}\tau^*(G)$. If a planar digraph G on n vertices has digirth at least g , then clearly $\tau^*(G) \leq \frac{n}{g}$ (this can be seen by assigning weight $1/g$ to each vertex). Therefore, a direct consequence of the conjecture of Goemans and Williamson would be that $f_g(n) \leq \frac{3n}{2g}$.

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